5. Series representation of Brownian motion. The "derivative of Brownian motion" is called white noise. Formally, take an ONB $\{\varphi_n\}$ of $L^2(I)$ and write $W'(t) = \sum_n X_n \varphi_n(t)$ where X_n are i.i.d $N(0,1)$, and then integrate term-by-term to write the expression $W(t) = \sum_n X_n \Phi_n(t)$ where $\Phi_n(t) = \int_0^t \varphi_n(s)ds$. What sense does this make? We *start with the series* $\sum_n X_n \Phi_n(t)$, and show that it converges (a.s.) in appropriate sense to a continuous function which can then be identified as Brownian motion. The Lévy construction given in the book essentially does this using the Haar basis (see Theorem 1.44 in [**MP**]).

Wiener himself used the Fourier basis to give a second construction of Brownian motion. The goal is to write $W(t) = \sum_{n=0}^{\infty} X_n \frac{\sqrt{2}\sin\pi(n+\frac{1}{2})t}{\pi(n+\frac{1}{2})}$. The difficulty is in showing that the series converges uniformly (a.s.) to a continuous function.

23 (Preliminary observations). Let $\psi_n(t) = \sqrt{2}\sin\pi(n+\frac{1}{2})t$. These are orthonormal in $L^2(I)$.

1. For any fixed $t \in I$, the series $\sum \frac{X_n}{\pi(n+\frac{1}{2})}\psi_n(t)$ converges almost surely.
2. For any $0 \le t_1 < \ldots < t_n \le 1$, the limit random variables $W(t_1), \ldots, W(t_n)$ are jointly normal with covariances $\sum \frac{1}{\pi^2(n+\frac{1}{2})^2}\psi_n(t)\psi_n(s)$ which is in fact equal to $t \wedge s$ (you do not need to show this last part).

24 (A lemma on trigonometric polynomials). Let $p(t) = \sum_{k=0}^{N-1} c_k \psi_k(t)$. Then, $\|p'\|_{\sup} \le 2\pi N^2 \|p\|_{\sup}$. In particular, $\int_0^1 (e^{2\lambda p(t)} + e^{-2\lambda p(t)})dt \ge \frac{1}{2\pi N^2}\exp\{\lambda \|p\|_{\sup}\}$ for any $\lambda > 0$.

25. Let $f(t) = \sum_{k=0}^{N-1} c_k X_k \psi_k(t)$ where X_k are i.i.d $N(0,1)$. Compute $\mathbf{E}[e^{\lambda f(t)}]$ and hence conclude that $\mathbf{P}\left(\|f\|_{\sup} > 4M\|c\|\right) \le 4\pi N^2 \exp\{-M^2\}$ where $\|c\|^2 = \sum_{k=0}^{N-1} c_k^2$.

26. Let $f_{k,\ell}(t) = \sum_{n=k}^{\ell-1} \frac{1}{\pi(n+\frac{1}{2})} X_n \psi_n(t)$. Use the previous problem to show that $\sum_k \|f_{2^k,2^{k+1}}\|_{\sup}$ is finite with probability one. Conclude that $W(t) := \sum_{n=0}^{\infty} X_n \frac{\sqrt{2}\sin\pi(n+\frac{1}{2})t}{\pi(n+\frac{1}{2})}$ is standard Brownian motion.

27 (Hölder continuity). For any $0 = N_0 < N_1 < N_2 < \ldots$ going to infinity and write $W = g_0 + g_1 + \ldots$ where $g_k(t) = f_{N_k, N_{k+1}}$. Let $\omega(h) = \sup_{|t-s|<h} |W_t - W_s|$.

1. For any $h > 0$, $\omega(h) \le h\|g_0'\|_{\sup} + 2\sum_{k=1}^{\infty} \|g_k\|_{\sup}$.
2. Choose N_k appropriately to show that $\omega(h) \le C\sqrt{h\log(1/h)}$ for some $C > 0$ (at least try to get $\omega(h) \le Ch^{\gamma}$ for any $\gamma < 1/2$).

By the same steps, one can show that Brownian bridge is represented by the random series $\sum_{n=1}^{\infty} X_n \frac{\sqrt{2}\sin(\pi n t)}{\pi n}$.